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Nonlinear gravity–capillary standing waves in water of arbitrary uniform depth

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Gravity–capillary standing waves in water of arbitrary uniform depth are considered. The classical perturbation calculation yields unbounded coefficients for some critical values of the depth. In the present paper solutions valid near the first critical value of the depth are derived. A problem of non-uniqueness is discovered and discussed. It is shown that two solutions exist, one with higher frequency and one with lower frequency than the zeroth-order solution. They are found analytically at the critical value of the depth and numerically in an interval around it. Graphs of the results are included.

1. Introduction

The problem of gravity standing waves in water of arbitrary uniform depth was solved to third order by Tadjbakhsh & Keller (1960). Their method was applied by Concus (1962) to solve the more general problem that includes capillary as well as gravitational forces.

These perturbation expansions were obtained by imposing a uniqueness condition which excludes certain fluid depths. Concus (1964) showed that the values of the depth excluded by this condition form a denumerably infinite set which is densely distributed over the entire positive real line. This difficulty raises theoretical questions about the validity of these expansions and about the mathematical existence of standing waves. These questions will not be answered in the present paper. However, we consider the solution obtained by Tadjbakhsh & Keller (1960) as a satisfactory third-order perturbation solution because it is defined for any value of the depth including those excluded by the uniqueness condition. These results are confirmed by the numerical calculations of Vanden-Broeck & Schwartz (1981).

The use of the uniqueness condition in the general problem with surface tension results in unbounded series coefficients for certain values of the depth (Concus 1962). Although these values of the depth were excluded by the uniqueness condition, the perturbation solution is clearly not satisfactory for values of the depth close to these critical values.

In the present paper we construct a perturbation solution valid at the first critical value of the depth. We show that two different solutions can exist at this critical value. These solutions are similar to the 'Wilton ripples' of the theory of gravity–capillary progressive waves (Wilton 1915; Pierson & Fife 1961; Schwartz & Vanden-Broeck 1979; Chen & Saffman 1979).

In addition we use the numerical scheme derived by Vanden-Broeck & Schwartz (1981) to compute the solution in the neighbourhood of the first critical value of the depth. We show that the two solutions obtained at the critical value are members of two different families of solutions.

We formulate the problem in §2. The main results obtained by Concus (1962) are summarized in §3. The perturbation solution valid at the first critical value is derived in §4. The numerical results are presented in §5.

2. Formulation

We consider the time-periodic two-dimensional potential flow of a fluid bounded below by a horizontal bottom and above by a free surface. We assume the motion to be periodic in the horizontal direction with wavelength λ. We measure lengths in units of $k^{-1} = \lambda/2\pi$.

Following Concus (1962), we define the parameters γ and δ by the relations

$$\gamma = \frac{\sigma k^2}{\rho g}, \tag{2.1}$$

$$\delta = \frac{\gamma}{1+\gamma}. \tag{2.2}$$

Here σ is the surface tension. For $\delta \ll 1$ the capillary effects are small, whereas for $1-\delta \ll 1$ they predominate.

We define Cartesian coordinates such that the motion is symmetric about the vertical plane $x = 0$ and such that $y = 0$ corresponds to the mean level. Let $k^{-1}h$ denote the mean depth, $[kg(1+\gamma)]^{\frac{1}{2}}\omega$ the angular frequency, $[kg(1+\gamma)]^{-\frac{1}{2}}\omega^{-1}t$ the time and a the amplitude of the linearized surface-wave motion. Then we define $\epsilon = ak$ and let $\epsilon k^{-1}\eta(x,t)$ denote the elevation of the free surface above the mean level and $\epsilon[g(1+\gamma)]^{\frac{1}{2}}k^{-\frac{3}{2}}\phi$ the velocity potential.

In dimensionless variables the motion of the fluid is described by the equations (see Concus 1962)

$$\Delta\phi = 0 \quad \text{in} \quad 0 < x < \pi, \quad -h < y < \epsilon\eta(x,t), \tag{2.3}$$

$$(1-\delta)\,\eta - \delta\eta_{xx}[1+\epsilon^2\eta_x^2]^{-\frac{3}{2}} + \omega\phi_t + \tfrac{1}{2}\epsilon(\phi_x^2+\phi_y^2) = 0 \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{2.4}$$

$$\phi_y = \omega\eta_t + \epsilon\phi_x\eta_x \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{2.5}$$

$$\partial\phi/\partial n = 0 \quad \text{on} \quad x = 0, \pi, \quad y = -h, \tag{2.6}$$

$$\eta_x = 0 \quad \text{on} \quad x = 0, \quad x = \pi, \tag{2.7}$$

$$\int_0^\pi \eta(x,t)\,\mathrm{d}x = 0, \tag{2.8}$$

$$\nabla\phi(x,y,t+2\pi) = \nabla\phi(x,y,t), \tag{2.9}$$

$$\int_{-h}^0\int_0^\pi\int_0^{2\pi} \phi(x,y,t)\sin t\cos x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2.10}$$

$$\int_{-h}^0\int_0^\pi\int_0^{2\pi} \phi(x,y,t)\cos t\cos x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = \tfrac{1}{2}\pi^2(\tanh h)^{\frac{1}{2}}. \tag{2.11}$$

As noted by Tadjbakhsh & Keller (1960) and Concus (1962), a unique solution does not exist for those values of h for which the frequency of the nth spatial harmonic $\{n[1+\delta(n^2-1)\tanh nh\}^{\frac{1}{2}}$ is an integral multiple of the fundamental frequency $(\tanh h)^{\frac{1}{2}}$: this yields the uniqueness condition

$$\frac{n[1+\delta(n^2-1)]\tanh nh}{\tanh h} \neq j^2 \quad (n = 2,3,\ldots;\ j = 1,2,\ldots). \tag{2.12}$$

3. Perturbation solution satisfying the uniqueness condition (2.12)

Following Tadjbakhsh & Keller (1960), Concus (1962) sought a solution as an expansion in powers of ϵ. Thus

$$\epsilon\eta = \epsilon\eta^0(x,t)+\epsilon^2\eta^1(x,t)+\tfrac{1}{2}\epsilon^3\eta^2(x,t)+O(\epsilon^4), \tag{3.1}$$

$$\epsilon\phi = \epsilon\phi^0(x,y,t)+\epsilon^2\phi^1(x,y,t)+\tfrac{1}{2}\epsilon^3\phi^2(x,y,t)+O(\epsilon^4), \tag{3.2}$$

$$\omega = \omega_0+\epsilon\omega_1+\tfrac{1}{2}\epsilon^2\omega_2+O(\epsilon^3). \tag{3.3}$$

The solution of the zeroth-order solution is given by

$$\eta^0 = \sin t\cos x, \tag{3.4}$$

$$\phi^0 = \frac{\omega_0}{\sinh h}\cos t\cos x\cosh(y+h), \tag{3.5}$$

$$\omega_0^2 = \tanh h. \tag{3.6}$$

This solution is made unique by imposing the condition (2.12).

Concus (1962) derived the following expressions for the first-and second-order solutions:

$$\eta^1 = \frac{1}{8}\left[\frac{\omega_0^2+\omega_0^{-2}}{1+3\delta}+\frac{\omega_0^{-2}-3\omega_0^{-6}}{1-3\delta\omega_0^{-4}}\cos 2t\right]\cos 2x, \tag{3.7}$$

$$\begin{aligned}\phi^1 = \beta_0&+\tfrac{1}{8}(\omega_0-\omega_0^{-3})\,t-\tfrac{1}{16}(3\omega_0+\omega_0^{-3})\sin 2t\\ &-\frac{3[\omega_0-2\delta\omega_0^{-3}-(1+2\delta)\,\omega_0^{-7}]}{16(1-3\delta\omega_0^{-4})\cosh 2h}\sin 2t\cos 2x\cosh(2y+h),\end{aligned} \tag{3.8}$$

$$\omega_1 = 0, \tag{3.9}$$

$$\eta^2 = b_{11}\sin t\cos x+b_{13}\sin 2t\cos 3x+b_{31}\sin 3t\cos x+b_{33}\sin 3t\cos 3x, \tag{3.10}$$

$$\begin{aligned}\phi^2 = \beta_2&+\beta_{13}\cos t\cos 3x\cosh 3(y+h)\\ &+\beta_{31}\cos 3t\cos x\cosh(y+h)+\beta_{33}\cos 3t\cos 3x\cosh 3(y+h),\end{aligned} \tag{3.11}$$

$$\omega_2 = \frac{1}{32}\frac{-2\omega_0^5-3(1+9\delta^2)\,\omega_0-3(4+6\delta-9\delta^2-27\delta^3)\,\omega_0^{-3}+9(1+5\delta+4\delta^2)\,\omega_0^{-7}}{(1+3\delta)\,(1-3\delta\omega_0^{-4})}, \tag{3.12}$$

where β_0 is an arbitrary constant. The constants b_{ij} and β_{ij} are defined by the relations (35) and (36) given by Concus (1962).

For $\delta = 0$ the solution (3.4)–(3.12) reduces to the solution given by Tadjbakhsh & Keller (1960). It can easily be checked that all the terms are bounded for any value of h if $\delta = 0$. Thus Tadjbakhsh & Keller's solution is a satisfactory third-order solution for any value of h.

For $\delta \neq 0$ some of the terms appearing in ω_2, η^1 and b_{33} are unbounded at the critical values of depth defined by the relations

$$1-3\delta\omega_0^{-4} = 0, \tag{3.13}$$

$$1-\delta(1+3\omega_0^{-4}) = 0. \tag{3.14}$$

These critical values correspond respectively to $n = j = 2$ and $n = j = 3$ in (2.12).

In §4 we derive a perturbation solution valid at the first critical value of the depth, i.e. at the value of the depth defined by (3.13).

4. Perturbation solution at the first critical value of the depth

We seek a perturbation solution of the form (3.1)–(3.3) valid when (3.13) is satisfied. We substitute the expansion (3.1)–(3.3) into the system of equations (2.3)–(2.11) and collect all terms of like powers of ϵ. The terms with ϵ to the first power in (2.4) and (2.5) are given by

$$(1-\delta)\,\eta^0-\delta\eta^0_{xx}+\omega_0\,\phi^0_t = 0 \quad \text{on} \quad y=0, \tag{4.1}$$

$$\phi^0_y-\omega_0\,\eta^0_t = 0 \quad \text{on} \quad y=0. \tag{4.2}$$

Equations (2.3) and (2.6)–(2.11) remain unchanged in form as equations for η^0, ϕ^0 and ω_0.

The terms of order ϵ^2 in (2.4), (2.5) and (2.11) are given by

$$(1-\delta)\,\eta^1-\delta\eta^1_{xx}+\omega_0\,\phi^1_t = F_0 \quad \text{on} \quad y=0, \tag{4.3}$$

$$\phi^1_y-\omega_0\,\eta^1_t = G_0 \quad \text{on} \quad y=0, \tag{4.4}$$

$$\int_{-h}^{0}\int_0^{\pi}\int_0^{2\pi}\phi^1\cos t\cos x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.5}$$

Here F^0 and G^0 are defined by

$$F^0 = -\tfrac{1}{2}[(\phi^0_x)^2+(\phi^0_y)^2]-\omega_0\,\eta^0\phi^0_{ty}-\omega_1\,\phi^0_t, \tag{4.6}$$

$$G^0 = \eta^0_x\,\phi^0_x-\eta^0\phi^0_{yy}+\omega_1\,\eta^0_t. \tag{4.7}$$

Equations (2.3) and (2.6)–(2.10) remain of the same form as equations in η^1, ϕ^1 and ω_1.

The solution of the zeroth-order problem defined by (2.3), (4.1), (4.2), (2.6)–(2.11) and (3.13) is

$$\eta^0 = \sin t\cos x+A\cos 2t\cos 2x, \tag{4.8}$$

$$\phi^0 = \frac{\omega_0}{\sinh h}\cos t\cos x\cosh(y+h)-\frac{A\omega_0}{\sinh 2h}\sin 2t\cos 2x\cosh 2(y+h), \tag{4.9}$$

$$\omega_0^2 = \tanh h. \tag{4.10}$$

Here A is an arbitrary constant. Thus the solution of the zeroth-order solution is not unique when (3.13) is satisfied.

Differentiating (4.3) with respect to t and substituting η^1_t from (4.4) and η^1_{xxt} from (4.4), after differentiating twice with respect to x, we obtain

$$-\delta\phi^1_{yxx}+(1-\delta)\,\phi^1_y+\omega_0^2\,\phi^1_t = H_0 \quad \text{on} \quad y=0. \tag{4.11}$$

Here H_0 is defined by

$$H_0 = \omega_0\,F^0_t+(1-\delta)\,G^0-\delta G^0_{xx}. \tag{4.12}$$

Separation of variables yields for the solution of (2.3) subject to (2.6)

$$\phi^1(x,y,t) = \sum_{n=0}^{\infty} A_n(t)\cos nx\cosh n(y+h). \tag{4.13}$$

Substituting (4.13) into (4.11), we obtain

$$\omega_0^2\cosh nh\,A_n''(t)+[(1-\delta)\,n+\delta n^3]\sinh nh\,A_n(t) = \frac{1}{\mu\pi}\int_0^{2\pi}H_0\cos nx\,\mathrm{d}x. \tag{4.14}$$

Here $\mu = 1$ for $n>0$ and $\mu = 2$ for $n=0$. Using (4.6)–(4.10), we can rewrite (4.14) in the form

$$\omega_0^2 A_0''(t) = \tfrac{1}{4}(3\omega_0^3+\omega_0^{-1})\sin 2t-2A^2(\omega_0^3\coth^2 2h+3\omega_0^3)\sin 4t, \tag{4.15}$$

$$\omega_0^2 \cosh h\, A_1''(t) + \sinh h\, A_1 = [2\omega_1 + \tfrac{1}{4}A(\omega_0^{-1} - 3\omega_0^3 + 4\omega_0 \coth 2h)] \cos t + \tfrac{1}{4}A[4\omega_0 \coth 2h + \omega_0^{-1} + 21\omega_0^3] \cos 3t, \quad (4.16)$$

$$\omega_0^2 \cosh 2h\, A_2''(t) + 2(1+3\delta) \sinh 2h\, A_2(t) = \{\tfrac{3}{4}[\omega_0^3 - (1+2\delta)\,\omega_0^{-1}] - 2A\omega_1 - 6A\omega_1\delta - 4A\omega_1\omega_0^2 \coth 2h\} \sin 2t, \quad (4.17)$$

$$\omega_0^2 \cosh 3h\, A_3''(t) + (3+5\delta) \sinh 3h\, A_3(t) = \tfrac{1}{4}A\omega_0 \cos t\,[(4+48\delta)\coth 2h - (3+24\delta)\,\omega_0^{-2} - 3\omega_0^2] - \tfrac{1}{4}A\omega_0 \cos 3t\,[(12+48\delta)\coth 2h + (3+24\delta)\,\omega_0^{-2} - 21\omega_0^2], \quad (4.18)$$

$$\omega_0^2 \cosh 4h\, A_4''(t) + (4+60\delta) \sinh 4h\, A_4(t) = A^2\omega_0 \sin 4t\,[(2+30\delta)\coth 2h + 2\omega_0 \coth^2 2h - 6\omega_0^2], \quad (4.19)$$

$$\omega_0^2 \cosh nh\, A_n''(t) + [(1-\delta)\,n + \delta n^3] \sinh nh\, A_n(t) = 0 \quad (n = 5, 6, \ldots). \quad (4.20)$$

From (2.9) and (4.13) it follows that A_n must be periodic in t with period 2π for $n \geqslant 1$ and from (3.13) and (4.20) that $A_n = 0$ for $n \geqslant 5$. The periodicity of A_1 requires the coefficient of $\cos t$ in (4.16) to be equal to zero. Thus

$$\omega_1 = \tfrac{1}{8}A(3\omega_0^3 - \omega_0^{-1} - 4\omega_0 \coth 2h). \quad (4.21)$$

If we set $A = 0$ in (4.15)–(4.21) we recover the system of equations derived by Concus (1962) for the first-order solution. In particular the solution of (4.17) is then given by

$$A_2 = -\frac{3[\omega_0 - 2\delta\omega_0^{-3} - (1+2\delta)\,\omega_0^{-7}]}{16(1-3\delta\omega_0^{-4}) \cosh 2h} \sin 2t. \quad (4.22)$$

This solution is unbounded since (3.13) is assumed to be satisfied. Therefore we do not set $A = 0$ in (4.15)–(4.21).

We shall determine the constant A in such a way that the solution of (4.17) is bounded. The appropriate compatibility condition is obtained by multiplying (4.17) by $\sin 2t$, integrating with respect to t from 0 to 2π, applying integration by parts twice to the term containing $A_2''(t)$, and using (3.13). Thus we find that the coefficient of $\sin 2t$ in the right-hand side of (4.17) must be equal to zero. This yields the relation

$$A\omega_1 = \frac{3[\omega_0^3 - (1+2\delta)\,\omega_0^{-1}]}{8 + 24\delta + 16\omega_0^2 \coth 2h}. \quad (4.23)$$

Substituting (4.21) into (4.23), we obtain

$$A = \pm\left\{\frac{3[\omega_0^3 - (1+2\delta)\,\omega_0^{-1}]}{[1 - 3\delta + 2\omega_0^2 \coth 2h]\,[3\omega_0^3 - \omega_0^{-1} - 4\omega_0 \coth 2h]}\right\}^{\frac{1}{2}}. \quad (4.24)$$

The remaining part of the calculation follows closely the work of Tadjbakhsh & Keller (1960) and Concus (1962). Integrating (4.15)–(4.19), we obtain

$$A_0 = -\tfrac{1}{16}(3\omega_0 + \omega_0^{-3}) \sin 2t + \tfrac{1}{8}A^2(\omega_0 \coth^2 2h + 3\omega_0) \sin 4t + \alpha_0 t + \beta_0, \quad (4.25)$$

$$A_1 = \frac{-A[4\omega_0 \coth 2h + \omega_0^{-1} + 21\omega_0^3]}{32 \sinh h} \cos 3t, \quad (4.26)$$

$$A_2 = \alpha_2 \sin 2t, \quad (4.27)$$

$$A_3 = \frac{A\omega_0 \cos t[(4+48\delta)\coth 2h-(3+24\delta)\,\omega_0^{-2}-3\omega_0^2]}{(12+20\delta)\sinh 3h-\omega_0^2\cosh 3h}$$

$$-\frac{A\omega_0 \cos 3t[(12+48\delta)\coth 2h+(3+24\delta)\,\omega_0^{-2}-21\omega_0^2]}{(12+20\delta)\sinh 3h-36\omega_0^2\cosh 3h}, \quad (4.28)$$

$$A_4 = \frac{A^2\omega_0 \sin 4t[(2+30\delta)\coth 2h+2\omega_0\coth^2 2h-6\omega_0^2]}{(4+60\delta)\sinh 4h-16\omega_0^2\cosh 4h}. \quad (4.29)$$

Here α_0, β_0 and α_2 are constants to be determined.

Substituting (4.13) into (4.3), we obtain

$$(1-\delta)\,\eta^1-\delta\eta^1_{xx} = F_0-\omega_0 \sum_{n=0}^{4} A'_n(t)\cos nx\cosh nh, \quad (4.30)$$

where F_0 and $A_n(t)$ are defined by (4.6) and (4.25)–(4.29). The function η_1 is therefore defined as the solution of (4.30) subject to (2.7).

The constant α_0 in (4.25) is evaluated by integrating (4.30) with respect to x between 0 and π and using (2.7) and (2.8). Thus we find

$$\alpha_0 = \tfrac{1}{8}\omega_0-\tfrac{1}{8}\omega_0^{-3}+\tfrac{1}{2}A^2\omega_0(1-\coth^2 2h). \quad (4.31)$$

This completes the determination of the first-order solution. It still contains an arbitrary constant α_2. This constant would be determined at second order in a way similar to the way that A was determined at first order. However, we shall not do this in this paper.

Equation (4.24) implies the existence of two solutions when (3.13) is satisfied. Relations (3.3) and (4.21) show that one solution is characterized by a frequency larger than the zeroth-order frequency, and the other solution by a frequency smaller. The wave profiles given by these two possibilities are illustrated in figure 1. These solutions are very similar to the 'Wilton ripples' of the theory of gravity–capillary progressive waves (Wilton 1915; Pierson & Fife 1961; Vanden-Broeck & Schwartz 1979; Chen & Saffman 1979).

In §5 we show that these two solutions are members of two different families of solution.

5. Numerical results

Concus' (1962) solution is satisfactory for values of the depth far enough away from the critical values (3.13) and (3.14). The solutions derived in §4 are correct at the critical value (3.13). Perturbation solutions valid for values of the depth near but not equal to the critical value (3.13) could be obtained by using the PLK method. An example of such a perturbation calculation can be found in Pierson & Fife (1961).

In the present work, we compute numerical solutions uniformly valid near the first critical value of the depth.

Vanden-Broeck & Schwartz (1981) derived a numerical scheme to compute pure gravity standing waves. Their numerical procedure is generalized to include the effect of surface tension by replacing their equation (2) by (2.4). The numerical procedure then follows closely the method outlined in §III of their paper.

Numerical values of ω as a function of $3\delta\omega_0^{-4}$ for $\epsilon = 0.005$ and $h = 3$ are shown in figure 2. These values were obtained with $N = 4$ in the equations (15) and (16) given by Vanden-Broeck & Schwartz (1981). No rigorous error bounds were calculated. However, the accuracy of the numerical results was estimated by increasing the value

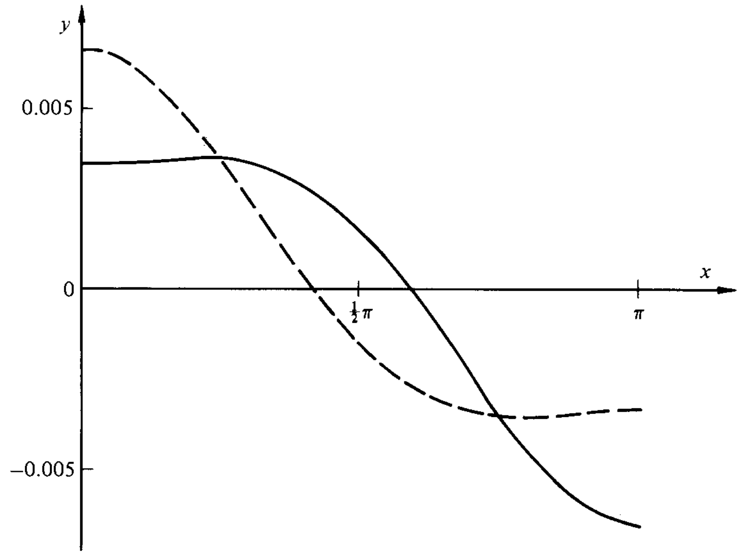

FIGURE 1. Profiles of the surface of the standing wave at $t = \frac{1}{2}\pi$. These curves are based on (4.8) with $\epsilon = 0.005$ and $h = 3$. The solid curve corresponds to $A > 0$ and the broken curves to $A < 0$.

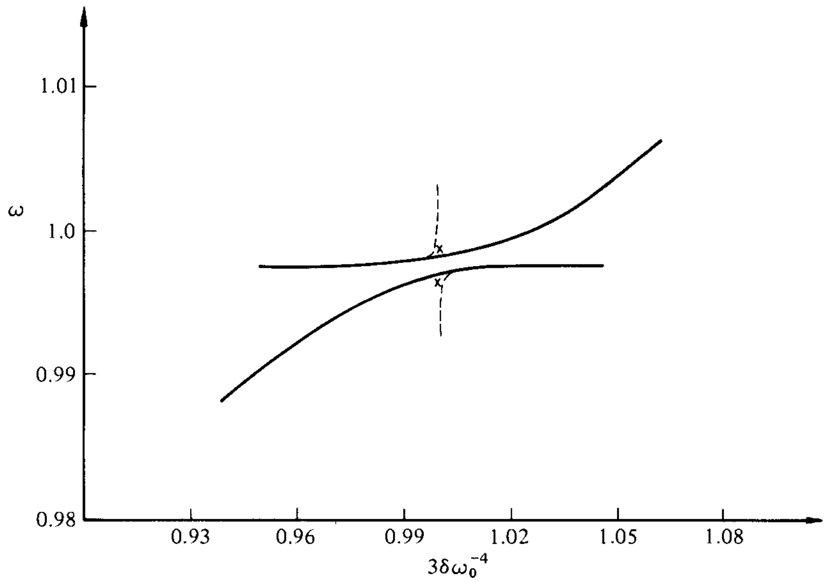

FIGURE 2. Values of ω as a function of $3\delta\omega_0^{-4}$ for $\epsilon = 0.005$ and $h = 3$. The solid curves correspond to the numerical computation, the broken curve to Concus' perturbation solution, and the two crosses to the solutions calculated in §4.

of N. The numerical values computed with $N > 4$ were found to be indistinguishable within graphical accuracy from those of figure 2 (for a detailed discussion of the convergence of the scheme as N is increased, see Vanden-Broeck & Schwartz 1981. Concus' perturbation solution for ω is represented by the broken line in figure 2. It is defined by (3.3), (3.6), (3.9) and (3.12). This solution is unbounded when $3\delta\omega_0^{-4} = 1$. The two crosses in figure 2 correspond to the perturbation solution of §4. They are defined by (3.3), (3.6), (4.23) and (4.24). These two solutions are in fair agreement with the numerical values. This constitutes a check on the validity of the numerical scheme.

The numerical results of figure 2 and similar results obtained for different values of the depth indicate that the solutions derived in §4 are members of two different families of solutions. One family of solutions agrees with Concus' perturbation solution for $3\delta\omega_0^{-4} < 1$ and the other family agrees with Concus' perturbation solution for $3\delta\omega_0^{-4} > 1$. Similar properties were found by Schwartz & Vanden-Broeck (1979) for gravity–capillary progressive waves in the neighbourhood of the first critical value of the capillary number.

Finally let us mention that Chen & Saffman (1979) have shown that the Wilton ripple phenomena is associated with a bifurcation in which a wave of permanent form can double its period. Similar results for standing waves would be obtained by defining the amplitude a as the coefficient of $\cos 2x \cos 2t$ in the expansion of the surface-wave elevation.

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